CONVERGENCE RATE FOR SEQUENCES OF MEASURABLE OPERATORS IN NONCOMMUTATIVE PROBABILITY SPACE

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Abstract: In this paper, we study the convergence rate for sequences of measurable operators under various conditions.
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1 Introduction

As it is well known, the law of large numbers (LLNs) is an essential theory in probability, statistics and related fields. In noncommutative probability, this issue was considered by some authors. Batty [2], Jaite [6], and Łuczak [9] proved some weak and strong laws of large numbers for sequences of successively independent measurable operators. Recently, Quang et al. [11] presented some strong laws of large numbers for sequences of positive measurable operators and applications. Other versions of LLNs can be found in [Quang et al. [12], Choi et al. [4], Klimczak [7]] and the references cited therein.

The convergence rate in noncommutative probability space have been established by several authors, e.g., Jajte [6], Götze and Tikhomirov [5], Chistyakov and Götze [3] and Stoica [13]. In particular, the authors in [3] gave estimates of the Lévy distance for freely independent partial sums and the author in [13] proved the Baum and Katz theorem in noncommutative Lorentz spaces.

In this paper, we present some results on convergence rate for sequences of measurable operators under various conditions.

2 Preliminaries

Let $\mathcal{A}$ be a von Neumann algebra (with unit $\mathbf{1}$) on a Hilbert space H and τ be a faithful normal tracial state on $\mathcal{A}$. A densely defined closed operator X in H is said to be *affiliated* to the von Neumann algebra $\mathcal{A}$ if U and the spectral projections of $|X|$ belong to $\mathcal{A}$, where $X = U|X|$ is the polar decomposition of X and $|X| = (X^*X)^{1/2}$. We notate $\widetilde{\mathcal{A}}$ for the set of operators which affiliated to the von Neumann algebra $\mathcal{A}$. An element of $\widetilde{\mathcal{A}}$ is called a *measurable operator*.

For notational consistency, $\widetilde{\mathcal{A}}$ will be denoted by $L^0(\mathcal{A},\tau)$. Then we have natural inclusions:

$$\mathcal{A} \equiv L^\infty(\mathcal{A},\tau) \subset L^q(\mathcal{A},\tau) \subset L^p(\mathcal{A},\tau) \subset ... \subset L^0(\mathcal{A},\tau) = \widetilde{\mathcal{A}}$$

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for $1 \leq p \leq q < \infty$, where $L^p(\mathcal{A}, \tau)$ is a Banach space of all elements in $L^0(\mathcal{A}, \tau)$ satisfying

$$||X||_p = [\tau(|X|^p)]^{\frac{1}{p}} < \infty.$$

For a set $\mathcal{S}$ of densely defined closed operators in H, $W^*(\mathcal{S})$ denotes the smallest von Neumann algebra to which each element of $\mathcal{S}$ is affiliated. For the case of $\mathcal{S} = \{X\}$ with a densely defined closed operator X, we write $W^*(X) \equiv W^*(\mathcal{S})$ for simple notation. $W^*(X)$ is said to be the von Neumann algebra generated by X.

Denote $e_B(X)$ by the spectral projection of the self-adjoint operator X corresponding to a Borel subset B of the real line $\mathbb{R}$. For two self-adjoint elements X and Y in $L^0(\mathcal{A}, \tau)$, we say that X and Y are *identically distributed* if $\tau(e_B(X)) = \tau(e_B(Y))$ for any Borel subset B of $\mathbb{R}$.

Let A_1 and A_2 be subalgebras of $\mathcal{A}$. Then we say that A_1 and A_2 are *independent* if

$$\tau(XY) = \tau(X)\tau(Y), \quad \forall X \in A_1, \forall Y \in A_2.$$

Two elements $X, Y \in L^0(\mathcal{A}, \tau)$ are said to be *independent* if the von Neumann algebras $W^*(X)$ and $W^*(Y)$ generated by X and Y, respectively, are independent.

A sequence $\{X_n, n \geq 1\} \subset L^0(\mathcal{A}, \tau)$ is said to be *pairwise independent* if, for all $m, n \in N$ and $m \neq n$, the algebras $W^*(X_m)$ and $W^*(X_n)$ are independent.

A sequence $\{X_n, n \geq 1\} \subset L^0(\mathcal{A}, \tau)$ is said to be *successively independent* if, for every n, the algebras $W^*(X_n)$ and $W^*(X_1, X_2, ..., X_{n-1})$ are independent. It is easily that the successively independence implies the pairwise independence.

Let $\{X_n, n \geq 1\}$ be a sequence in $L^0(\mathcal{A}, \tau)$ and $X \in L^0(\mathcal{A}, \tau)$. We say that the sequence $\{X_n, n \geq 1\}$ *converges in measure* to X, denoted by $X_n \xrightarrow{\tau} X$ *as* $n \to \infty$ if, for any $\epsilon > 0$, $\tau\left[e_{(\epsilon,\infty)}(|X_n - X|)\right] \to 0$ as $n \to \infty$.

For further information about the theory of noncommutative probability we refer to (Jajte [6], Nelson [10], Yeadon [15]).

For convenience, from now until the end of the paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

3 Main results

In this section we establish some results on convergence rate for sequence of measurable operators.

The following theorem is a noncommutative version of Proposition 2.4 in Li and Hu [8].

Theorem 3.1. *Let $s > 0$ and let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent measurable operators satisfying*

$$\sum_{n=1}^{\infty} \frac{\tau\left(|X_n - \tau(X_n)|^2\right)}{n^s} < \infty. \tag{3.1}$$

Put $S_n = \sum_{k=1}^{n} X_k$*, then for any* $\varepsilon > 0$*,*

$$\sum_{n=1}^{\infty} n^{1-s}\tau\left(e_{[\varepsilon;\infty)}\left|\frac{S_n - \tau(S_n)}{n}\right|\right) < \infty. \tag{3.2}$$

If $\{X_n, n \geq 1\}$ *is a sequence of successively independent measurable operators satisfying* (3.1)*, then for any* $\varepsilon > 0$*, there exists a sequence of projections* q_n *in* $\mathcal{A}$ *such that*

$$\sum_{n=1}^{\infty} n^{1-s}\tau(q_n) < \infty, \quad \text{and} \quad \left\|\big(S_n - \tau(S_n)\big)(\mathbf{1} - q_n)\right\|_{\infty} \leq n\varepsilon. \tag{3.3}$$

Proof. For any $\varepsilon > 0$, by Chebyshev's inequality and (3.1), we get

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{1-s}\tau\left(e_{[\varepsilon;\infty)}\left|\frac{S_n - \tau(S_n)}{n}\right|\right) &= \sum_{n=1}^{\infty} n^{1-s}\tau\Big[e_{[n\varepsilon,\infty)}\left(|S_n - \tau(S_n)|\right)\Big] \\
&\leq \sum_{n=1}^{\infty} n^{1-s}\frac{\tau\left(|S_n - \tau(S_n)|\right)}{(n\varepsilon)^2} \\
&= \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{1+s}}\tau\left(\left|\sum_{k=1}^{n}\left(X_k - \tau(X_k)\right)\right|^2\right) \\
&= \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{1+s}}\sum_{k=1}^{n}\tau\left(|X_k - \tau(X_k)|^2\right) \\
&= \frac{1}{\varepsilon^2}\sum_{k=1}^{\infty}\tau\left(|X_k - \tau(X_k)|^2\right)\sum_{n=k}^{\infty}\frac{1}{n^{1+s}} \\
&\leq C\sum_{k=1}^{\infty}\frac{\tau\left(|X_k - \tau(X_k)|^2\right)}{k^s} < \infty.
\end{aligned}$$

Hence (3.1) holds. Since $\{X_n, n \geq 1\}$ is a sequence of successively independent measurable operators, by Kolmogorov's inequality, we have, for any $\varepsilon > 0$, there exists a sequence of projections q_n in $\mathcal{A}$ such that

$$\tau(q_n) \leq \frac{1}{(n\varepsilon)^2}\sum_{k=1}^{n}\tau\left(|X_k - \tau(X_k)|^2\right),$$

and

$$\left\|\big(S_n - \tau(S_n)\big)(\mathbf{1} - q_n)\right\|_{\infty} \leq n\varepsilon.$$

Thus,

$$\sum_{n=1}^{\infty} n^{1-s}\tau(q_n) < \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{s+1}}\sum_{k=1}^{n}\tau\left(|X_k - \tau(X_k)|^2\right) < \infty.$$

□

The following theorem is an extension of Lemma 2.1 from Bai, Chen and Sung [1] to noncommutative probability.

Theorem 3.2. *Let* $1 \le p \le 2, \alpha \le 0$ *and let* $\{X_n, n \ge 1\}$ *be a sequence of pairwise independent measurable operators with* $\sum_{n=1}^{\infty} n^{\alpha-p+1}\tau(|X_n|^p) < \infty$. *Then for all* $\epsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha}\tau\left(e_{[\epsilon,\infty)}\left(\left|\frac{1}{n}\sum_{k=1}^{n}\Big(X_k - \tau(X_k)\Big)\right|\right)\right) < \infty.$$

Proof. For each $n \ge 1$, put

$$X_k^{(n)} = X_k e_{[0,n)}\left(|X_k|\right), \quad S_n = \frac{1}{n}\sum_{k=1}^{n} X_k, \quad \widetilde{S}_n = \frac{1}{n}\sum_{k=1}^{n} X_k^{(n)}, \quad M_n = \tau(S_n), \quad \widetilde{M}_n = \tau(\widetilde{S}_n).$$

Then, for any $\gamma > 0$, we have

$$p \equiv e_{[2\gamma,\infty}\left(|S_n - M_n|\right) \wedge e_{[0,\gamma)}\left(\left|\widetilde{S}_n - \widetilde{M}_n\right|\right) \wedge \bigwedge_{k=1}^{n} e_{[0,n)}\left(|X_k|\right) = 0.$$

Indeed, if there exists h of norm one, $h \in p(H)$, then $h \in e_{[0,n)}\left(|X_k|\right)(H)$ and, consequently, $X_k(h) = X_k.e_{[0,n)}\left(|X_k|\right)(h) = X_k^{(n)}(h)$, for all $k = 1, 2, ..., n$, which yields $S_n(h) = \widetilde{S}_n(h)$, and $M_n(h) = \widetilde{M}_n(h)$.

Thus, from the elementary properties of the spectral decomposition, we obtain

$$\begin{aligned}
2\gamma = 2\gamma||h||_{\infty} &\le \left\| \, |S_n - M_n| \, e_{[2\gamma,\infty)}\left(|S_n - M_n|\right)(h)\right\|_{\infty} = \left\| \left(S_n - M_n\right)(h)\right\|_{\infty} \\
&\le \left\|(S_n - \widetilde{S}_n)(h)\right\|_{\infty} + \left\|(\widetilde{S}_n - \widetilde{M}_n)(h)\right\|_{\infty} + \left\|(\widetilde{M}_n - M_n)(h)\right\|_{\infty} \\
&= \left\|(\widetilde{S}_n - \widetilde{M}_n)(h)\right\|_{\infty} \\
&= \left\||\widetilde{S}_n - \widetilde{M}_n| e_{[2\gamma,\infty)}\left(|\widetilde{S}_n - \widetilde{M}_n|\right)(h)\right\|_{\infty} \\
&\le \gamma||h||_{\infty} = \gamma,
\end{aligned}$$

which is impossible, so $p = 0$ and this implies

$$e_{[2\gamma,\infty)}\left(|S_n - M_n|\right) \prec e_{[\gamma,\infty)}\left(\left|\widetilde{S}_n - \widetilde{M}_n\right|\right) \vee \Big(\bigvee_{k=1}^{n} e_{[n,\infty)}(|X_k|)\Big).$$

Using the pairwise independence of the sequence $\{X_n, n \ge 1\}$ and Chebyshev's inequality, we obtain that

$$\begin{aligned}
\tau\Big(e_{[2\gamma,\infty)}(|S_n - M_n|)\Big) &\le \tau\Big(e_{[\gamma,\infty)}(|\widetilde{S}_n - \widetilde{M}_n|)\Big) + \sum_{k=1}^{n} \tau\Big(e_{[n,\infty)}(|X_k|)\Big) \\
&\le \frac{1}{\gamma^2}\tau\Big(|\widetilde{S}_n - \widetilde{M}_n|^2\Big) + \sum_{k=1}^{n} \tau\Big(e_{[n,\infty)}(|X_k|)\Big) \\
&\le \frac{1}{\gamma^2}\sum_{k=1}^{n} \tau(|X_k^{(n)}|^2) + \sum_{k=1}^{n} \tau\Big(e_{[n,\infty)}(|X_k|)\Big) \\
&\le \frac{1}{\gamma^2}\sum_{k=1}^{n} \tau(|X_k^{(n)}|^2) + \sum_{k=1}^{n} n^{-p}\tau\big(|X_k|^p\big).
\end{aligned}$$

Now, take any $\epsilon > 0$, with $\gamma = \dfrac{n\epsilon}{2}$, we get

$$\tau\Big(e_{[\epsilon,\infty)}(|S_n - M_n|)\Big) \le \frac{4}{\epsilon^2 n^2}\sum_{k=1}^{n} \tau(|X_k^{(n)}|^2) + \sum_{k=1}^{n} n^{-p}\tau\big(|X_k|^p\big).$$

Since

$$\begin{aligned}
\tau\big(|X_k^{(n)}|^2\big) &= \int_0^n \lambda^2 \tau\Big(e_{d\lambda}(|X_k|)\Big) \\
&= \int_0^n \lambda^p \lambda^{2-p} \tau\Big(e_{d\lambda}(|X_k|)\Big) \\
&\le n^{2-p}\int_0^\infty \lambda^p \tau\Big(e_{d\lambda}(|X_k|)\Big) \\
&= n^{2-p}\tau\left(|X_k|^p\right).
\end{aligned}$$

We have

$$\tau\Big(e_{[\epsilon,\infty)}(|S_n - M_n|)\Big) \le \frac{4}{\epsilon^2 n^2}\sum_{k=1}^{n} n^{2-p}\tau(|X_k|^p) + \sum_{k=1}^{n} n^{-p}\tau\big(|X_k|^p\big) \le C\sum_{k=1}^{n} n^{-p}\tau\big(|X_k|^p\big),$$

which implies that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^\alpha \tau\Big(e_{[\epsilon,\infty)}(|S_n - M_n|)\Big) &\le C\sum_{n=1}^{\infty} n^{\alpha-p}\sum_{k=1}^{n} \tau\big(|X_k|^p\big) \\
&\le C\sum_{n=1}^{\infty}\sum_{k=1}^{n} k^{\alpha-p+1}\tau\big(|X_k|^p\big)
\end{aligned}$$

(Because $n^{\alpha-p} \le k^{\alpha-p+1}$, for all $1 \le k \le n$)

$$\le C\sum_{k=1}^{\infty} k^{\alpha-p+1}\tau\big(|X_k|^p\big) < \infty.$$

□

In Theorem 3.2, if we put $\alpha = -1$, then we have the following corollary which is a noncommutative version of Lemma 2.1 in Bai, Chen and Sung [1].

Corollary 3.3. *Let $1 \le p \le 2$ and let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent measurable operators with $\sum_{n=1}^{\infty} n^{-p}\tau(|X_n|^p) < \infty$. Then for all $\epsilon > 0$*

$$\sum_{n=1}^{\infty} n^{-1}\tau\left(e_{[\epsilon,\infty)}\left(\left|\frac{1}{n}\sum_{k=1}^{n}\Big(X_k - \tau(X_k)\Big)\right|\right)\right) < \infty.$$

Taking $\alpha = 0$ in Theorem 3.2, we have the following corollary which is connected with the study of weak law of large numbers (see Corollary 3.5).

Corollary 3.4. *Let $1 \le p \le 2$ and let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent measurable operators with $\sum_{n=1}^{\infty} n^{-p+1}\tau(|X_n|^p) < \infty$. Then for all $\epsilon > 0$*

$$\sum_{n=1}^{\infty} \tau\left(e_{[\epsilon,\infty)}\left(\left|\frac{1}{n}\sum_{k=1}^{n}\Big(X_k - \tau(X_k)\Big)\right|\right)\right) < \infty. \tag{3.4}$$

Corollary 3.5. *Let $1 \le p \le 2$ and let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent measurable operators with $\sum_{n=1}^{\infty} n^{-p+1}\tau(|X_n|^p) < \infty$. Then for all $\epsilon > 0$*

$$\frac{1}{n}\sum_{k=1}^{n}\Big(X_k - \tau(X_k)\Big) \xrightarrow{\tau} 0 \ \text{ as } \ n \to \infty.$$

Proof. By (3.4), we have for any $\varepsilon > 0$,

$$\tau\left(e_{[\epsilon,\infty)}\left(\left|\frac{1}{n}\sum_{k=1}^{n}\Big(X_k - \tau(X_k)\Big)\right|\right)\right) \to 0 \ \text{ as } \ n \to \infty.$$

□

The following theorem is a noncommutative version of Theorem 2.1 in [14].

Theorem 3.6. *Let $\{X, X_n, n \ge 1\}$ be a pairwise independent sequence of identically distributed measurable operators and let $\{a_n, n \ge 1\}$ be a sequence of positive constants with $a_0 = 0$, $\dfrac{a_n}{n} \uparrow$.*

If $\sum_{n=1}^{\infty} \tau\left[e_{(a_n,\infty)}(|X|)\right] < \infty$, then for all $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} n^{-1}\tau\left[e_{(a_n\epsilon,\infty)}\left(\left|S_n - \tau(\widetilde{S}_n)\right|\right)\right] < \infty,$$

where

$$S_n = \sum_{i=1}^{n} X_i, \quad \widetilde{S}_n = \sum_{i=1}^{n} X_i e_{[0,a_n]}(|X_i|).$$

Proof. Put $Y_i = X_i e_{[0,a_n]}(|X_i|)$, give $\gamma > 0$, we have

$$p \equiv e_{(\gamma,\infty)}\big(|S_n - \tau(\widetilde{S}_n)|\big) \wedge e_{[0,\frac{\gamma}{2}]}\big(|\widetilde{S}_n - \tau(\widetilde{S}_n)|\big) \wedge \left(\bigwedge_{i=1}^{n} e_{[0,a_n]}(|X_i|)\right) = 0$$

(the proof is the same as that of Theorem 3.2 and is omitted).
This yields

$$e_{(\gamma,\infty)}\big(|S_n - \tau(\widetilde{S}_n)|\big) \prec e_{(\frac{\gamma}{2},\infty)}\big(|\widetilde{S}_n - \tau(\widetilde{S}_n)|\big) \vee \left(\bigvee_{i=1}^{n} e_{(a_n,\infty)}(|X_i|)\right).$$

It follows that

$$\tau\Big[e_{(\gamma,\infty)}\big(|S_n - \tau(\widetilde{S}_n)|\big)\Big] \le \tau\Big[e_{(\frac{\gamma}{2},\infty)}\big(|\widetilde{S}_n - \tau(\widetilde{S}_n)|\big)\Big] + \sum_{i=1}^{n}\tau\Big[e_{(a_n,\infty)}(|X_i|)\Big].$$

For $\varepsilon > 0$, by taking $\gamma = a_n\varepsilon$ and using Chebyshev's inequality, we obtain that

$$\begin{aligned}\tau\Big[e_{(a_n\varepsilon,\infty)}\big(|S_n - \tau(\widetilde{S}_n)|\big)\Big] &\le \tau\Big[e_{(\frac{a_n\varepsilon}{2},\infty)}\big(|\widetilde{S}_n - \tau(\widetilde{S}_n)|\big)\Big] + \sum_{i=1}^{n}\tau\Big[e_{(a_n,\infty)}(|X_i|)\Big] \\ &\le \frac{4}{\varepsilon^2 a_n^2}\tau\Big(\Big|\widetilde{S}_n - \tau(\widetilde{S}_n)\Big|^2\Big) + \sum_{i=1}^{n}\tau\Big[e_{(a_n,\infty)}(|X_i|)\Big].\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1}\tau\left[e_{(a_n\epsilon,\infty)}\left(\Big|S_n - \tau(\widetilde{S}_n)\Big|\right)\right] &\le \frac{4}{\varepsilon^2 a_n^2}\sum_{n=1}^{\infty} n^{-1}\tau\Big(\Big|\widetilde{S}_n - \tau(\widetilde{S}_n)\Big|^2\Big) + \sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n}\tau\Big[e_{(a_n,\infty)}(|X_i|)\Big] \\ &\le \frac{4}{\varepsilon^2}\sum_{n=1}^{\infty} n^{-1}a_n^{-2}\tau\Big(\Big|\widetilde{S}_n - \tau(\widetilde{S}_n)\Big|^2\Big) + \sum_{n=1}^{\infty}\tau\Big[e_{(a_n,\infty)}(|X|)\Big] \\ &:= \frac{4}{\varepsilon^2}I_1 + I_2.\end{aligned}$$

Since $I_2 < \infty$ by the assumption, it remains to show that $I_1 < \infty$. Using the pairwise independence of the sequence $\{X, X_n, n \ge 1\}$, we get

$$\begin{aligned}I_1 &= \sum_{n=1}^{\infty} n^{-1}a_n^{-2}\left\{\sum_{i=1}^{n}\tau\left(\big|Y_i - \tau(Y_i)\big|^2\right) + \sum_{i\ne j}\left[\tau(Y_i^*Y_j) - \tau(Y_i^*)\tau(Y_j)\right]\right\} \\ &= \sum_{n=1}^{\infty} n^{-1}a_n^{-2}\sum_{i=1}^{n}\tau\left(\big|Y_i - \tau(Y_i)\big|^2\right) \le \sum_{n=1}^{\infty} n^{-1}a_n^{-2}\sum_{i=1}^{n}\tau\left(|Y_i|^2\right) \\ &= \sum_{n=1}^{\infty} n^{-1}a_n^{-2}\sum_{i=1}^{n}\tau\left[|X_i|^2 e_{[0,a_n]}(|X_i|)\right] = \sum_{n=1}^{\infty} a_n^{-2}\tau\left[|X|^2 e_{[0,a_n]}(|X|)\right].\end{aligned}$$

Noting that the condition $\frac{a_n}{n} \uparrow$ implies

$$\sum_{n=i}^{\infty} \frac{1}{a_n^2} \leq \frac{i^2}{a_i^2} \sum_{n=i}^{\infty} \frac{1}{n^2} \leq \frac{2i}{a_i^2}.$$

Therefore, we have

$$\begin{aligned} I_1 &\leq \sum_{n=1}^{\infty} a_n^{-2} \sum_{i=1}^{n} \tau\left[|X|^2 e_{[a_{i-1},a_i]}(|X|)\right] = \sum_{i=1}^{\infty} \tau\left[|X|^2 e_{[a_{i-1},a_i]}(|X|)\right] \sum_{n=i}^{\infty} \frac{1}{a_n^2} \\ &\leq 2\sum_{i=1}^{\infty} \tau\left[|X|^2 e_{[a_{i-1},a_i]}(|X|)\right] \frac{i}{a_i^2} \leq 2\sum_{i=1}^{\infty} i\tau\left[e_{[a_{i-1},a_i]}(|X|)\right] \\ &\leq 2\sum_{i=0}^{\infty} \tau\left[e_{(a_i,\infty)}(|X|)\right] < \infty. \end{aligned}$$

$\square$

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TÓM TẮT

TỐC ĐỘ HỘI TỤ ĐỐI VỚI DÃY CÁC TOÁN TỬ ĐO ĐƯỢC TRONG KHÔNG GIAN XÁC SUẤT KHÔNG GIAO HOÁN

Trong bài báo này, chúng tôi nghiên cứu tốc độ hội tụ của dãy các toán tử đo được với những điều kiện khác nhau.

Từ khóa: Tốc độ hội tụ; toán tử đo được; đại số von Neumann.